1202 Summer 2008: Solutions

1 H is a subgroup of G if (A) $e \in H$, (B) $g, h \in H \Rightarrow gh \in H$ and (C) $g \in H \Rightarrow g^{-1} \in H$.

(i) $G = \mathbb{R}$ under addition, $H = \{x \in G : x \geq 0\}$. $2 \in H$, but the inverse of 2, which is -2, is not in H, so H is not a subgroup.

(ii) $G = \mathbb{R}$ under addition, $H = \mathbb{Z}$.

(A) The identity of G is 0 and $0 \in \mathbb{Z}$.

(B) Let $h, k \in \mathbb{Z}$. Then $h + k \in \mathbb{Z}$.

(C) Let $h \in \mathbb{Z}$. Then the inverse of h, which is $-h$ is in $\mathbb{Z}$.

Thus H is a subgroup.

(iii) $G = S_5$, $H = \{g \in G : g^3 = e\}$. Clearly any 3-cycle is in H, so $h = (1\ \ 2\ \ 3) \in H$ and $k = (1\ \ 2\ \ 4) \in H$. But

$hk = (1\ \ 2\ \ 3)(1\ \ 2\ \ 4) = (1\ \ 3)(2\ \ 4)$

has order 2 and so $(hk)^3 \neq e$ and hence $hk \notin H$. Thus H is not a subgroup.

(iv) G is any abelian group, $H = \{g \in G : g^3 = e\}$.

(A) $e^3 = e$, so $e \in H$.

(B) Suppose $h, k \in H$, so $h^3 = k^3 = e$. Then, because G is abelian, $(hk)^3 = h^3k^3 = ee = e$ and hence $hk \in H$.

(C) Suppose $h \in H$, so $h^3 = e$. Then $(h^{-1})^3 = h^{-3} = (h^3)^{-1} = e^{-1} = e$ and hence $h^{-1} \in H$.

Thus H is a subgroup.

(v) G is any group, K is a subgroup of G and $a \in G$, and $H = \{a^{-1}ka : k \in K\}$.

(A) $a^{-1}ea = a^{-1}a = e$, so $e \in H$

(B) Suppose $g, h \in H$, say $g = a^{-1}ka$ and $h = a^{-1}ja$, where $k, j \in K$. Then $gh = (a^{-1}ka)(a^{-1}ja) = a^{-1}kja$. Since K is a subgroup, $kj \in K$ and hence $gh \in H$.

(C) Suppose $g \in H$, say $g = a^{-1}ka$ for some $k \in K$. $g^{-1} = (a^{-1}ka)^{-1} = a^{-1}k^{-1}a$. Since K is a subgroup, $k^{-1} \in K$ and so $g^{-1} \in H$.

Thus H is a subgroup.

Definition; similar examples to (i) - (iv) seen; (v) unseen

2. (a) (a) Lagrange's Theorem says that if G is a finite group and H a subgroup, then $|H|$ divides $|G|$.

Let $g \in G$. If g has infinite order, then $e, g, g^2, \ldots$ are all distinct elements of G, contradicting the fact that G is finite. Hence g has finite order, say m. Then $\{e, g, \ldots, g^{m-1}\}$ are all distinct and hence form a subgroup of G of order m. By Lagrange's Theorem, m divides $|G|$

(b) $\mathbb{Z}_p^*$ is a group of order $p-1$. Hence every element of $\mathbb{Z}_p^*$ has order dividing $p-1$, say $p-1 = o(a)k$ and hence $\overline{a}^{p-1} = \overline{a}^{o(a)k} = \overline{1}^k = \overline{1}$.

(c) In $\mathbb{Z}_{19}^*$, $\overline{3}^{18} = \overline{1}$ by (b). Hence $\overline{3}^{1799} = \overline{3}^{18\times 100-1} = \overline{3}^{-1}$. But by inspection $3 \times 6 = 18 \equiv -1$; hence $\overline{3}^{-1} = -\overline{6} = \overline{13}$ and hence $\overline{3}^{1799} = \overline{13}$.

(d) Suppose $\overline{x}^3 = \overline{5}$. Then $\overline{x}^{27} = \overline{5}^9 = e$, since order of $\overline{5}$ is 9. But also $\overline{x}^{18} = e$ by Fermat's Little Theorem. Hence $\overline{x}^9 = \overline{x}^{27-18} = e$. But then $\overline{5}^3 = \overline{x}^9 = e$ and hence $\overline{5}$ has order dividing 3, a contradiction.

(a),(b) Bookwork + (c) similar example seen, (d) unseen

3. (a) $\det A = \sum_{\sigma \in S_n} sign(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \ldots a_{n,\sigma(n)}$

(b) The (i,j)-*minor* M_{ij} of A is the determinant of the $n-1 \times n-1$ matrix obtained by deleting the i^{th} row and j^{th} column of A.

(c) The (i,j)-*cofactor* C_{ij} of A is $(-1)^{i+j} M_{ij}$.

(d) The *adjugate* of A, denoted $adj(A)$ is the matrix whose (i,j) entry is C_{ji}

(b) We use the following facts:

(i) Formula for expansion along ith row: $\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$.

(ii) A matrix with two rows the same has determinant zero.

Now the (i,i) entry of $Aadj(A)$ is given by

$\sum_{j=1}^{n} a_{ij} (adjA)_{ji} = \sum_{j=1}^{n} a_{ij} C_{ij} = \det(A)$.

Now consider the (1,2)-entry. This is given by

$\sum_{j=1}^{n} a_{1j} (adjA)_{j2} = \sum_{j=1}^{n} a_{1j} C_{2j}$

But this last is just the expansion along the second row of the determinant of

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

which is zero, since the matrix has two rows the same. Thus the (1,2)-entry of $Aadj(A)$ is zero: an exactly similar argument applies to the (r,s)-entry for $r \neq s$.

Thus $Aadj(A)$ is a diagonal matrix with diagonal entries all $\det A$, i.e. $Aadj(A) = \det(A) I_n$.

(c) $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$

Matrix of minors $= \begin{pmatrix} a^2 - bc & ca - b^2 & c^2 - ab \\ ab - c^2 & a^2 - bc & ac - b^2 \\ b^2 - ac & ab - c^2 & a^2 - bc \end{pmatrix}$

Matrix of cofactors $= \begin{pmatrix} a^2 - bc & b^2 - ac & c^2 - ab \\ c^2 - ab & a^2 - bc & b^2 - ac \\ b^2 - ac & c^2 - ab & a^2 - bc \end{pmatrix}$

$adj(A) = \begin{pmatrix} a^2 - bc & c^2 - ab & b^2 - ac \\ b^2 - ac & a^2 - bc & c^2 - ab \\ c^2 - ab & b^2 - ac & a^2 - bc \end{pmatrix}$

Hence $A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{a^3+b^3+c^3-3abc} \begin{pmatrix} a^2 - bc & c^2 - ab & b^2 - ac \\ b^2 - ac & a^2 - bc & c^2 - ab \\ c^2 - ab & b^2 - ac & a^2 - bc \end{pmatrix}$

valid when $a^3 + b^3 + c^3 - 3abc \neq 0$.

(a),(b) Bookwork (c) Similar examples seen

4. (a) (i) λ is an *eigenvalue* of A if there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

(ii) $\mathbf{v}$ is an *eigenvector* of A if $\mathbf{v} \neq \mathbf{0}$ and there exists $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

(iii) The *eigenspace* $E_\lambda = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \lambda\mathbf{v}\}$.

(iv) The *characteristic polynomial* $c_A(t)$ of A is given by $\det(tI - A)$

(v) A is *diagonalizable* (over $\mathbb{R}$) if there exists $P \in GL_n(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

Basic criterion A is diagonalizable if and only if there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors.

(b) Suppose $\sum_i \mathbf{u}_i = \mathbf{0}$ where not all $\mathbf{u}_i = \mathbf{0}$. Then pick a shortest non-trivial relation that holds; by renumbering, we can assume the relation is

$$\sum_{i=1}^p \mathbf{u}_i = 0 \qquad (1)$$

where each $\mathbf{u}_i \neq \mathbf{0}$.

Then $A\sum_{i=1}^p \mathbf{u}_i = \mathbf{0}$, so

$$\sum_{i=1}^p \lambda_i \mathbf{u}_i = 0 \qquad (2)$$

Now taking eqn (2) from λ_p times equation (1) we get

$$\sum_{i=1}^{p-1} (\lambda_p - \lambda_i)\mathbf{u}_i = 0 \qquad (3)$$

and this is a shorter relation than (1) (since it involves at most $p-1$ terms) and is also non-trivial (since the terms are non-zero since $\lambda_p \neq \lambda_i$). This is a contradiction.

So the only possibility is that there are no such relations, i.e. the sum is direct.

Now pick a basis $\mathcal{B}_i$ for each E_{λ_i}. By Lemma 4.21, $\mathcal{B} = \cup_{i=1}^r \mathcal{B}_i$ is a basis for $\oplus_i E_{\lambda_i}$, which is of dimension $\sum_i e_i = n$. Hence $\oplus_i E_{\lambda_i} = \mathbb{R}^n$ and hence $\mathbb{R}^n$ has a basis $\mathcal{B}$ consisting of eigenvectors, so A is diagonalizable by the basic criterion.

(c) Let $A = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

Then $c_A(t) = \det \begin{pmatrix} t-3 & 0 & -1 & 0 \\ 0 & t-3 & 0 & -1 \\ 0 & 0 & t-4 & 0 \\ 0 & 0 & 0 & t-4 \end{pmatrix} = (t-3)^2(t-4)^2$

$$E_3 = \{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : (A - 3I)\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}$$

$$= \{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}$$

and this is clearly 2-dimensional.

Similarly E_4 is 2-dimensional.

Thus $\dim(E_3) + \dim(E_4) = 2 + 2 = 4 = n$ so by (b) A is diagonalizable.

(a),(b) Bookwork (c) examples seen

5. (i) $A = \begin{pmatrix} 1 & -5 \\ 2 & 8 \end{pmatrix}$

$c_A(t) = \det \begin{pmatrix} t-1 & 5 \\ -2 & t-3 \end{pmatrix} = t^2 - 9t + 18 = (t-3)(t-6)$. Hence eigenvalues of A are 3 and 6.

$\lambda = 3$; then eigenvector is solution to $\begin{pmatrix} 2 & 5 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$, e.g. $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

Similarly $\lambda = 6$ yields eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Hence if we take $P = \begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix}$, then $P^{-1}AP = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$ is diagonal.

(ii) $A^n = P(P^{-1}AP)^n P^{-1} = \begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 6^n \end{pmatrix} (1/3) \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} =$

$= (1/3) \begin{pmatrix} 5.3^{n+1} & -6^n \\ -2.3^n & 6^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 5.3^n - 2.6^n & 5.3^n - 5.6^n \\ -2.3^n + 2.6^n & -2.3^n + 5.6^n \end{pmatrix}$

(iii) Write $\mathbf{v}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$. Then we can write the equations as $\mathbf{v}_{n+1} = A\mathbf{v}_n$. It is then clear that the solution is $\mathbf{v}_n = A^n \mathbf{v}_0$, so

$\mathbf{v}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 5.3^n - 2.6^n & 5.3^n - 5.6^n \\ -2.3^n + 2.6^n & -2.3^n + 5.6^n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$= \begin{pmatrix} 5.3^n - 2.6^n & 5.3^n - 5.6^n \\ -2.3^n + 2.6^n & -2.3^n + 5.6^n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 5.3^n - 2.6^n \\ -2.3^n + 2.6^n \end{pmatrix}$

Similar calculations seen

6. (a) Let λ be an eigenvalue of A (in $\mathbf{C}$) with corresponding eigenvector $\mathbf{v}$ (in $\mathbf{C}^n$), so

$$A\mathbf{v} = \lambda\mathbf{v}$$

Taking the complex conjugate and transposing, we get

$$\bar{\mathbf{v}}^T A = \bar{\lambda}\mathbf{v}^T$$

Now we have

$$\bar{\lambda}\mathbf{v}^T\mathbf{v} = \bar{\mathbf{v}}^T A\mathbf{v} = \bar{\mathbf{v}}^T\lambda\mathbf{v} = \lambda\bar{\mathbf{v}}^T\mathbf{v}$$

so

$$(\bar{\lambda} - \lambda)\mathbf{v}^T\mathbf{v} = 0$$

Write $\mathbf{v} = (v_1 \quad v_2 \quad \dots \quad v_n)$, where $v_j = a_j + ib_j$ $(a_j, b_j \in \mathbb{R})$. Since $\mathbf{v} \neq \mathbf{0}$, at least one a_j or $b_j > 0$. Now

$$\bar{\mathbf{v}}^T\mathbf{v} = \sum_{j=1}^n v_j\bar{v_j} = \sum_{i=1}^n (a_j^2 + b_j^2) > 0$$

and hence $\bar{\mathbf{v}}^T\mathbf{v} \neq 0$. Hence

$$\lambda = \bar{\lambda}$$

i.e. $\lambda \in \mathbb{R}$.

(b) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$.

$$c_A(t) = \det\begin{pmatrix} t-1 & -2 & -1 \\ -2 & t-4 & -2 \\ -1 & -2 & t-1 \end{pmatrix} = (t-1)^2(t-4) - 8 - 8(t-1) - (t-4)$$

$= t^3 - 6t^2 + 9t - 4 - 9t + 4 = t^3 - 6t^2 = t^2(t-6)$.

Hence eigenvalues are 0,0,6.

Find corresponding eigenspaces

$$E_0 = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here the RRE form is $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and so y and z are the free variable, and $x = -2y - z$, so the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus E_0 has basis

$\{\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}.$

Apply G-S: $\mathbf{u}_1 = \frac{1}{|\mathbf{v}_1|}\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$

$\mathbf{w}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ -\frac{2}{5} \\ 1 \end{pmatrix}$. Normalizing $\mathbf{u}_2 = \frac{1}{\sqrt{30}}\begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}.$

$E_6 = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} -5 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

Row reduce:

$$\begin{pmatrix} -5 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 12 & -24 \\ 0 & -6 & 12 \\ 1 & 2 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 2 & -5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

This has basis $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and normalising we get $\mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$

Thus suitable orthogonal matrix is given by $P = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{5} & -2/\sqrt{30} & 2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{pmatrix}.$

(a) Bookwork (b) Similar examples seen